Grid-Free Particle Method Applied to the Equations of Unsteady Compressible Fluid Motion

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I. Introduction

PARTICLE methods have been used for the study of various phenomena, such as clustering of galaxies, flow of electrons, and motion of incompressible inviscid fluids.[1] In these methods, the movements of the particles, which possess quantifiable properties (mass, charge, and vorticity, respectively, for the above-mentioned phenomena), are determined by particular physical laws or equations of motion and are traced by a Lagrangian scheme. This scheme is efficient because it does not require the grid generation that is indispensable for Eulerian schemes, e.g., finite-difference methods, and finite-element methods. An additional advantage is that, in most cases, the quantity of mass, total vorticity, and some of the other properties are constant in time. Furthermore, computation is simplified because the positions of the particles are calculated by ordinary differential equations.

In the field of computational fluid dynamics, the particle models known as the vortex methods have been applied and developed independently of similar models in other fields. Though having the merits of the particle methods mentioned before, the vortex methods also have some restrictions, i.e., fluid is inviscid and incompressible. The reason for the inviscid restriction was that "adding the viscous term $\nu\Delta\omega$ is not convenient in a Lagrangian reference frame because it involves derivatives with respect to the Eulerian Coordinates."[2] However, investigators like Ogami and Akamatsu[3] and Chorin[4] have succeeded in treating the viscous term in a Lagrangian scheme by introducing diffusion velocity and random walks, respectively. The diffusion velocity method gives solutions that are smoother than the random vortex method. Also, this method is more dependent on the Reynolds number than the conventional vortex methods. Nevertheless, reasonable solutions are obtained even when $Re = 0$.

Until recently, numerical analysis of unsteady compressible fluids has been performed only by the Eulerian methods. There has been some work that combines an Eulerian method to a Lagrangian formulation, as in the work of Van Roessel and Hui.[5] However, the restriction of incompressibility on the vortex methods still remains. Considering this, we present a grid-free numerical method for the application of the particle model to the compressible fluids. To show the validity of our model, we focus on the time-dependent one-dimensional phenomena, which are brought about by solving the linear wave equation, the Burgers equation, and the equation for compressible fluids, both with and without a diffusion term.

II. Description of the Method

Ogami and Akamatsu[3] treated the viscous term in the particle scheme in a deterministic way by introducing the diffusion velocity. This velocity is defined to be proportional to the density gradient and the diffusion coefficient and inversely proportional to the density. The total velocity induced on each particle is the summation of the convection velocity, which is derived from the convection term of the Navier-Stokes equation, and the diffusion velocity, which is from the diffusion term of the equation.

In the rest of this section, we will give a brief description of how the particle method can be modified to approximate the Burgers equation and the equations of finite waves both with and without the diffusion term. The key to the method is to define the velocity for the Burgers equation or the acceleration for the equation of finite waves that are induced on each particle by the other particles. The definitions for velocity and acceleration are then used to simulate the equations of motion applied to compressible fluids.

A. Particle Method Applied to Burgers Equation

First, consider a function $F(x, t)$ moving at a velocity $U(x, t)$ in the x direction. This motion is described by the following equation in conservation form:

$$\frac{\partial F}{\partial t} + \frac{\partial (UF)}{\partial x} = 0 \tag{1}$$

In this form, the time-dependent velocity at which the function $F(x, t)$ moves appears in the second term of the equation.

Following this idea, consider the following Burgers equation, which describes the motion of finite waves in a compressible fluid[6]:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2} \tag{2}$$

This can be transformed into the following equation in conservation form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[\left(\frac{\rho}{2} - \frac{\nu}{\rho}\frac{\partial \rho}{\partial x}\right)\rho\right] = 0 \tag{3}$$

Comparing Eq. (1) and Eq. (3), we notice that the density ρ is carried in the x direction with a velocity

$$u_d = \frac{\rho}{2} - \frac{\nu}{\rho}\cdot\frac{\partial \rho}{\partial x} \tag{4}$$

To approximate the solution to Eq. (2), we first discretize the initial condition $\rho(x, 0) = \rho_0(x)$ into N particles, each with a Gaussian distribution for the core function. The center of these particles moves with the velocity given by Eq. (4). With this distribution, the density at the center of the jth particle x_j is given by

$$\rho(x_j) = \sum_{i=1}^{N} \frac{\rho_0 \Delta x}{\sqrt{\pi}\sigma} \exp\left[\frac{-(x_i - x_j)^2}{\sigma^2}\right] \tag{5}$$

where σ is the shape parameter that determines the Gaussian distribution, ρ_0 the initial density distribution, and Δx the initial distance between the particles. The magnitude of each particle is initially given by $\rho_0 \Delta x$. It is important to note that the dimension of ρ is not necessarily that of mass-per-unit-volume but depends on the equations. For instance, the dimension of ρ is LT^{-1} for the Burgers equation.

Using Eq. (5), we obtain the derivative of ρ as

$$\frac{\partial \rho(x_j)}{\partial x_j} = \frac{2\rho_0 \Delta x}{\sqrt{\pi}\sigma^3} \sum_{i=1}^{N} (x_i - x_j) \exp\left[\frac{-(x_i - x_j)^2}{\sigma^2}\right] \tag{6}$$

Thus, the velocity induced on the jth particle is

$$u_j = \frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{\rho(x_j)}{2} - \frac{\nu}{\rho(x_j)}\frac{\partial \rho(x_j)}{\partial x_j} \tag{7}$$

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To obtain an approximate solution to the Burgers equation, we solve the ordinary differential Eq. (7) for each of the N particles, where the motion of the particle is given by

$$x_j^{k+1} = x_j^k + u_j^k \Delta t \tag{8}$$

where Δt is the time step and x_j^k and u_j^k the position and velocity, respectively, of the jth particle at time $k\Delta t$.

B. Particle Method Applied to the Equation for a Compressible Fluid

The equation for a compressible fluid with a diffusion term is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{a^2}{\rho}\cdot\frac{\partial \rho}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{9}$$

where u describes the fluid velocity and a the speed of sound. Using γ for the ratio of the specific heats and a_0 for the stagnation speed of sound, the relation $a = a_0(\rho/\rho_0)^{(\gamma-1)/2}$ is given.[6]

First, consider the equation for a compressible fluid without the diffusion term:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{a^2}{\rho}\cdot\frac{\partial \rho}{\partial x} = 0 \tag{10}$$

Equation (10) is equivalent to the following ordinary differential equation:

$$\frac{\mathrm{D}u_j}{\mathrm{D}t} = \frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} = -\frac{a^2}{\rho(x_j)}\cdot\frac{\partial \rho(x_j)}{\partial x_j} \tag{11}$$

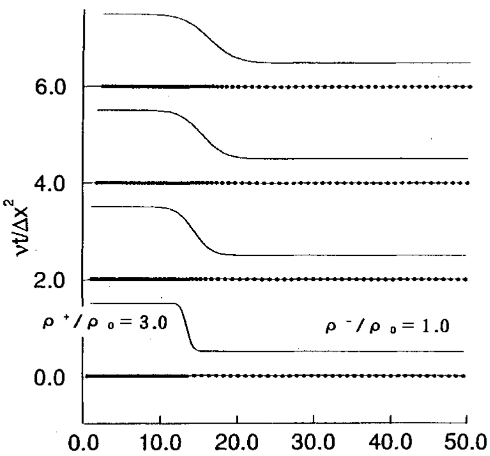

Fig. 1 Relaxation of the discontinuity of the density distribution.

This equation indicates that the particle at the position x_j is given the acceleration that is proportional to the density gradient and the square of the sound velocity and inversely proportional to the density [when the left-hand side of Eq. (11) is not an acceleration but a velocity, we can treat the motion equation as the diffusion[3]]. Consequently, Eq. (10) for a compressible fluid is reduced to the ordinary differential Eq. (11).

Similarly, the equation for a compressible fluid with the diffusion term,[6] Eq. (9), reduces to the following ordinary differential equation:

$$\frac{\mathrm{D}u_j}{\mathrm{D}t} = \frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} = -\frac{a^2}{\rho(x_j)}\cdot\frac{\partial \rho(x_j)}{\partial x_j} + \nu\frac{\partial^2 u}{\partial x^2} \tag{12}$$

To our knowledge, there is no satisfactory way to compute the term $\partial^2 u/\partial x^2$ as precisely as the density distribution by the particle scheme. One obvious approximation is to calculate this term using finite differencing. However, since the points are generally not evenly distributed, we chose to calculate this term using the spline function interpolation, where the second derivative of this spline function is then used to approximate $\partial^2 u/\partial x^2$.

III. Results

A. Burgers Equation

In this section, the particle method is applied to the Burgers equation, Eq. (2), to simulate the density discontinuity that approaches the steady shock wave. As shown in Fig. 1, at time $t = 0$, the particles are arranged ⅓Δx apart for $x/\Delta x \leq 14$ and Δx apart for $x/\Delta x \geq 14$. The resulting density ratio is $\rho/\rho_0 = 3$ for $x/\Delta x < 14$ and $\rho/\rho_0 = 1$ for $x/\Delta x > 14$. Other parameters used in these calculations are $\Delta x/\sigma = 1$ and $\nu(\Delta t)/(\Delta x^2) = 0.2$. The slope at the discontinuity cannot be infinite because the core of the particles is finite. For time $t > 0$, the particles move at the speed and in the direction given by Eq. (7), and, as a result, the density variation changes. As time increases, the slope at the discontinuity decreases. However, it does not reach zero but approaches the steady solution. This is because the nonlinear term and the diffusion term are balanced in time.

B. Equation for a Compressible Fluid

The equation for a compressible fluid, Eq. (11), reduces to the linear wave equation $\partial^2\rho/\partial t^2 = a^2(\partial^2\rho/\partial x^2)$ when the variation of the density is small. The sound speed a in this case is a constant. The particle method used to simulate a head-on collision of two linear waves is presented in this section.

Initially, at time $t = 0$, 61 particles are placed along the x axis with distance Δx apart. The numerical parameters used are $\Delta x/\sigma = 0.5$ and $a(\Delta t/\Delta x) = 1$. Small waves are created by giving small movements to the particles at the left ($x = 0$) and

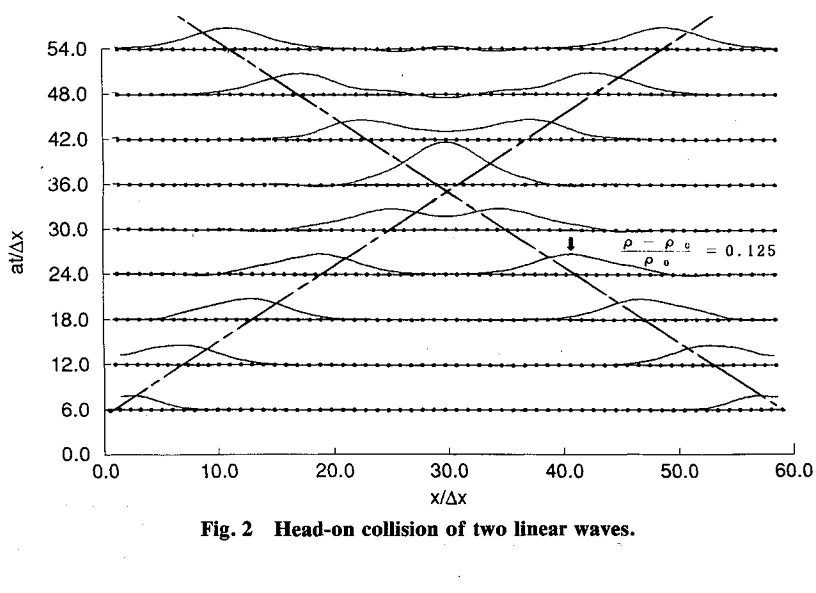

Fig. 2 Head-on collision of two linear waves.

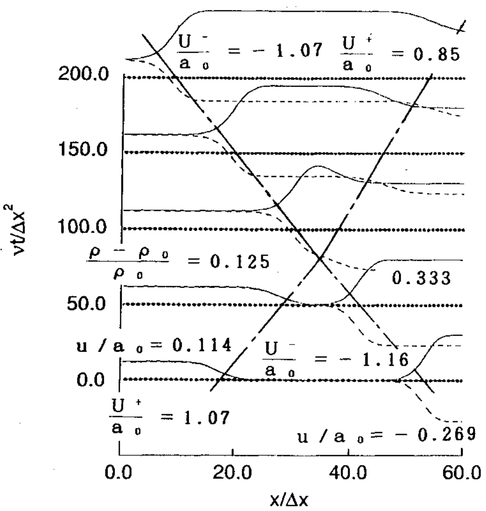

Fig. 3 Head-on collision of two shock waves.

right ($x = 60$) ends as shown in Fig. 2. The movement δ given to the particle at the left end is

$$\delta = \frac{\Delta x}{2}\left(1 - \cos\frac{2\pi}{30} i\right) \qquad i = 1, \ldots, 15 \tag{13}$$

where i indicates the time $t = i\Delta t$ and $\delta = \Delta x$ when $i \geq 16$. The movement of the particle at the right end is given by $-\delta$. As a result of these movements, two waves are generated. In other words, a finite slope of the density distribution is created by the movements added to the particles at the ends so that finite acceleration given by Eq. (11) is induced on the rest of the particles. Consequently, the density variation, i.e., $(\rho - \rho_0)/\rho_0$, of amplitude 0.125 travels from one particle to another, although each particle moves slightly around the initial position.

As depicted by the loci of the waves in Fig. 2 (indicated by dash-dotted lines), the two waves created at the ends travel at a constant speed toward the opposite ends. The waves merge at the center of the x axis, at around time 36. At this time, the amplitude of the waves is doubled, and the distance between the particles at the center becomes the smallest. After this, the wave separates into the original waves, and they travel to opposite ends. As can be observed, this collision does not affect the shape of the waves or the speed at which they travel.

C. Equation for a Compressible Fluid with the Diffusion Term

In this section, we consider application of the particle method to the equation for finite waves with the diffusion term, Eq. (9). At time $t = -50$, the particles are placed at a distance of $0.8\Delta x$ apart for $-80 \leq x \leq -20$, $1.0\Delta x$ apart for $-20 \leq x \leq 60$, and $0.6\Delta x$ apart for $60 \leq x \leq 120$ (Fig. 3). As a result, the density distribution is discontinuous. The variation of the density distribution $(\rho - \rho_0)/\rho_0$ is depicted by a solid line. The dashed line and the dash-dotted line indicate the particle velocity and the locus of the shock wave, respectively. The values of the other parameters used in this calculation are $\nu(\Delta t/\Delta x^2) = 0.5$, $\nu/(a_0\Delta x) = 5.0$, $\Delta x/\sigma = 1.0$, and $\gamma = 1.4$. As shown in Fig. 3, the particles accelerated according to Eq. (12) generate a head-on collision of two shock waves of different magnitude. The shapes of the shock waves vary when the two waves are merged; however, they recover after separation. Because of this collision, the velocity of the shock wave traveling to the right changes from $U^+/a_0 = 1.07$ to 0.85 and the velocity of the shock wave traveling to the left changes from $U^-/a_0 = -1.16$ to -1.07. These values agree with the analytic solutions.[6,7]

IV. Conclusion

A grid-free numerical method is presented for solving the equations of compressible fluids by using the particle model. This model is used to simulate numerically the time-dependent one-dimensional phenomena that arise by solving the linear wave equation, the Burgers equation, and the equation for compressible fluids, both with and without the diffusion term. The formation of the steady shock wave and the interactions of the linear waves and of the nonlinear shock waves are successfully simulated by our method. The ability of the method to treat the compressibility has been adequately shown. As far as we know, this is the first successful simulation of the compressible fluids by the particle scheme.

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Vortex-Induced Energy Separation in Shear Flows

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I. Introduction

ONE of the interesting features of unsteady aerodynamics is its wealth of counterintuitive phenomena. An excellent example of these is that of energy separation in a flow, where regions of high and low energy can be identified in the flowfield. A measure of the energy of the flow is the total temperature T_t.

In adiabatic, inviscid flows that do no mechanical shaft work, we are used to thinking of the total temperature as being constant, since the energy of the flow is conserved. However, T_t is constant along a streamline only for steady flows; in unsteady flows, the T_t along the fluid particle trajectory can vary significantly. This phenomenon is reflected in the following energy equation:

$$C_p \frac{DT_t}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} \tag{1}$$

We see that if the flow is steady, then the T_t is constant along the streamline; but if an unsteady pressure field is imposed on the flow, then the T_t of the flow can change both in space and time. In the latter case, we can say that the energy of the flow becomes separated; some regions of the flowfield contain more energy than others.

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